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NUMERICAL ANALYSIS OF AXISYMMETRIC BUCKLING OF PLATES UNDER RADIAL COMPRESSION

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Nonlinear boundary-value problems of axisymmetric buckling of simply supported and clamped plates under radial compression are formulated for a system of six first-order ordinary differential equations with independent fields of finite displacements and rotations. Multivalued solutions are obtained by the shooting method with specified accuracy. Bifurcation of the solutions of the problem is studied, and a parametric bifurcation diagram is constructed for various values of the loading parameter. Curves of buckling modes are given for three branches of the solution. The numerical results agree with available theoretical data.

Key words: *axisymmetric buckling, plate, numerical analysis.*

Introduction. The problem of buckling of a circular plate under uniform radial compression is one of the well-known problems of elasticity theory, which has long attracted the attention of both mechanical engineers and mathematicians. Like the Euler problem of column buckling, it is a classical problem of bifurcation theory for solutions of nonlinear boundary-value problems. The studies addressing this problem can be classified into two groups: analytical methods and theoretical analysis. Bryan [1] and Dinnik [2] obtained exact analytical solutions to linearized problems of axisymmetric buckling of simply supported and clamped plates, treated as stability problems of the plane state of a plate. These solutions, expressed in terms of Bessel functions, determine a discrete spectrum of eigenvalues λ_n ($n \in N$) for the loading parameter λ. Moreover, Bryan [1] and Nadai [3] obtained a discrete spectrum of a linearized problem for asymmetric (oscillating along the circumferential coordinate) buckling modes. The first eigenvalue for these modes was found to be much higher than that for the axisymmetric modes. In view of this fact, the problem of axisymmetric buckling of a plate is still of interest to researches.

Friedrichs and Stoker [4] were the first to formulate a nonlinear problem of axisymmetric buckling of a plate under uniform radial compression using the von Kármán fourth-order equations. Using the power series method they obtained solutions for a simply supported plate that bifurcate from the first eigenvalue λ_1 of the linearized problem. Other boundary-value problems in this formulation were later solved invoking direct methods [5].

Friedrichs and Stoker [6] pioneered the theoretical analysis of nonlinear problems of axisymmetric buckling of plates under radial compression. It has been shown [6] that for a simply supported plate in the interval $\lambda_1 < \lambda \leqslant \lambda_2$, only one (mirror-symmetrical) pair of buckling modes exists. Vorovich [7] substantiated the method of linearizing nonlinear problems of shell buckling and showed that in a plate problem, the parameter λ is a bifurcation point of the solution if and only if it is equal to the eigenvalue of the linearized problem. It has been shown [8–10] that for simply supported and clamped plates for all $\lambda > \lambda_n$, n pairs of axisymmetric buckling modes with $n-1$ internal nodes exist. The main results of [8–10] can be found in [11], and those of [7] are included in [12].

Recently, this class of nonlinear problems has been the subject of extensive theoretical investigations. However, insufficient attention has been given to numerical solutions of these problems. Nontrivial solutions of large norm were obtained in [13] by simplifying the mathematical formulation of the problem: the fourth-order boundary-value problem was reduced to an inadequate boundary-value problem for the Bessel equation for the function $\sin\theta$ (θ is the local angle of rotation of the radial element of the plate).

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In the present paper, the nonlinear problem of axisymmetric buckling of a plate under radial compression is solved in a refined formulation using six first-order nonlinear equations which describe arbitrary angles of rotation and take into account transverse shear strains.

System of Equations. In cylindrical coordinates (r, φ, z), a circular plate is defined by the equation

$$r = Rt \quad \forall t \in [0, 1], \quad \forall \varphi \in [0, 2\pi], \quad \forall z \in [-h, h],$$

where t is an independent variable, R is the radius of the plate, and $2h$ is the plate thickness. We consider the case where the middle surface of the deformed plate remains axisymmetric and is described by the parametric equations

$$r = Ry_2(t), \qquad z = Ry_3(t) \quad \forall t \in [0, 1], \quad \forall \varphi \in [0, 2\pi],$$

where y_2 and y_3 are the unknown coordinates of a point.

To study the axisymmetric deformation of the plate, we use the equations of the nonlinear shell model with independent fields of finite displacements and rotations [14]. The plate material is considered transversely isotropic and linearly elastic. The original system of equations comprises the constitutive relations

$$\begin{gathered} U_{11} = (1-\nu^2)F^{-1}T_{11} - \nu U_{22}, \qquad V_{11} = (1-\nu^2)H^{-1}M_{11} - \nu V_{22}, \\ U_{13} = \gamma F^{-1}T_{13}, \qquad T_{22} = \nu T_{11} + FU_{22}, \qquad M_{22} = \nu M_{11} + HV_{22}, \end{gathered} \tag{1}$$

the kinematic relations

$$\begin{gathered} RV_{11} = \theta', \qquad RV_{22} = t^{-1}\sin\theta, \qquad U_{22} = t^{-1}(y_2 - t), \\ y_2' = (1+U_{11})\cos\theta + U_{13}\sin\theta, \qquad y_3' = -(1+U_{11})\sin\theta + U_{13}\cos\theta, \end{gathered} \tag{2}$$

and the static equations

$$\begin{gathered} (tT_1)' - T_{22} + RtP_1 = 0, \quad (tT_3)' + RtP_3 = 0, \quad (tM_{11})' - M_{22}\cos\theta - RtT_{13} + RtQ_2 = 0, \\ T_{11} = T_1\cos\theta - T_3\sin\theta, \qquad T_{13} = T_1\sin\theta + T_3\cos\theta. \end{gathered} \tag{3}$$

In (1)–(3), $F = 2hE$, $3H = 2h^3E$, $\gamma = E/G$, E is Young's modulus, G is the transverse shear modulus, ν is Poisson's ratio, $\theta(t)$ is the unknown angle of rotation of the local basis about the basis of the cylindrical coordinate system, $U_{iJ}(t)$, $V_{ii}(t)$, $T_{iJ}(t)$, and $M_{ii}(t)$ are the components of the metric and flexural strains, forces, and moments in the rotated basis, respectively ($i = 1, 2$ and $J = 1, 2, 3$), $T_1(t)$, $T_3(t)$, $P_1(t)$, $P_3(t)$, and $Q_2(t)$ are the components of the forces, surface loads, and moments in the basis of the cylindrical coordinate system, respectively, and the prime denotes differentiation with respect to t.

Rotation independent of displacements produces transverse shear strains whose measure is the metric component $U_{13}(t)$ related to the shear force $T_{13}(t)$ by the constitutive relation (1).

Equations (1)–(3) reduce to the system

$$\begin{gathered} y_0' = t^{-1}[(1-\nu^2)y_1 - \nu\sin y_0], \qquad y_1' = t^{-1}(\nu y_1 + \sin y_0)\cos y_0 + \varepsilon^{-1}f_3 - tq_2, \\ y_2' = \varepsilon\gamma t^{-1}f_3\sin y_0 + (1+\varepsilon f_1)\cos y_0, \qquad y_3' = \varepsilon\gamma t^{-1}f_3\cos y_0 - (1+\varepsilon f_1)\sin y_0, \\ y_4' = t^{-1}[\nu f_2 + \varepsilon^{-1}(y_2 - t)] - tp_1, \qquad y_5' = -tp_3, \\ f_1 = t^{-1}[(1-\nu^2)f_2 - \varepsilon^{-1}\nu(y_2 - t)], \quad f_2 = y_4\cos y_0 - y_5\sin y_0, \quad f_3 = y_4\sin y_0 + y_5\cos y_0 \end{gathered} \tag{4}$$

for six unknown functions $y_0 = \theta$, $y_1 = tM_{11}R/H$, $y_2 = r/R$, $y_3 = z/R$, $y_4 = tT_1/C$, and $y_5 = tT_3/C$ and the parameters $\varepsilon^2 = h^2/(3R^2)$, $p_J = P_JR/C$, $q_2 = Q_2R^2/H$, and $C = \varepsilon F$.

System (4) describes the nonlinear bending of a circular plate for specified values of the loading parameters p_1, p_3, and q_2 and the stiffness parameters ε, γ, and ν and for specified boundary conditions. This system is singular for the variable t at the point $t = 0$. For $\gamma = 0$, system (4) formulates the problem of strong bending of the plate without transverse shear strains (the Kirchhoff model for large rotations).

For system (4), we formulate two boundary-value problems of axisymmetric deformation of a circular plate under uniform radial compressive loading of intensity P. Since a surface load is absent, one should set $q_2 = p_3 = p_1 = 0$ in system (4). At the pole $t = 0$, the symmetry and regularity conditions $y_0(0) = 0$, $y_2(0) = 0$, and $[t^{-1}y_5(t)]_{t\to 0} \to 0$ should hold. In the numerical analysis, however, these conditions do not hold because of the

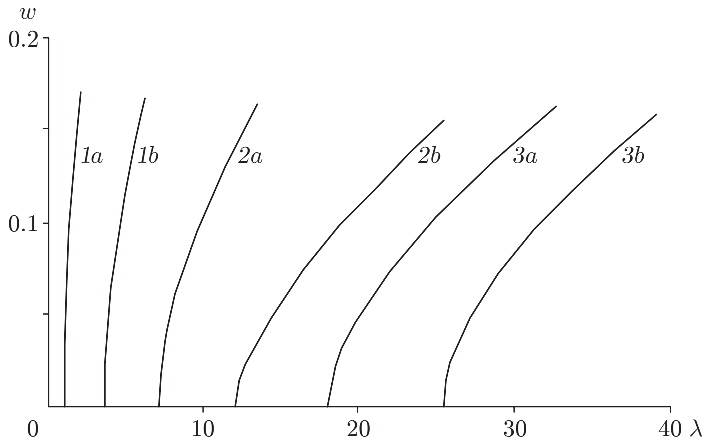

Fig. 1

presence of polar singularities in system (4). Therefore, these conditions are replaced by the following asymptotically exact conditions at a point $t = \delta$ close to the pole:

$$(\nu - 1)y_1 + \sin y_0 = 0, \quad (\nu - 1)(y_4 \cos y_0 - y_5 \sin y_0) + \varepsilon^{-1}(y_2 - \delta) = 0, \quad y_5 = 0. \tag{5}$$

By virtue of conditions (5), the equalities $T_{22} = T_{11}$, $M_{22} = M_{11}$, and $T_3 = 0$, which should hold at the pole, are satisfied at this point.

At the boundary point $t = 1$, the following conditions are specified: $y_1(1) = 0$, $y_3(1) = 0$, and $y_4(1) = -p$ for a movable, simply supported contour (problem A) or $y_0(1) = 0$, $y_3(1) = 0$, $y_4(1) = -p$ ($p = P/C$) for a movable, clamped contour (problem B). From Eqs. (4) and the boundary conditions follows the obvious result $y_5(t) \equiv 0$ ($T_3 \equiv 0$), which was not used in the numerical algorithm but was invoked to control the stability of the solution against small perturbations of the boundary parameters.

The nonlinear problems A and B were solved by the shooting method from the point $t = 1$ to the point $t = \delta$. The boundary conditions (5) form a system of implicit equations for additional initial parameters of the shooting method. Bifurcation of the solutions of the boundary-value problems was determined by varying the loading parameter. Numerical implementation of the algorithm was performed with the Mathcad software package.

Numerical Results. The solutions of the boundary-value problems A and B for an isotropic plate with parameters $\nu = 0.25$, $\gamma = 2.5$, and $\varepsilon = 0.025$ are presented in tables and figures. We introduce the following notation: $\lambda = P/P_{1a}$ is the normalized loading parameter, $P_{1a} = Cp_{1a}$ is the first bifurcation value of the compressive force in the simply supported case, $\theta(1)$ is the boundary value of the angle of rotation, $u = 1 - y_2(1) = 1 - r(1)/R$ and $w = |y_3(\delta)| = |z(\delta)|/R$ are the parameters of the radial and axial displacements of the boundary points, respectively, $\tau_i = T_{ii}/C$ and $\mu_i = M_{ii}R/H$ are the parameters of the internal forces and moments, respectively, and $\mu(\delta) = \mu_1(\delta) = \mu_2(\delta)$.

The following three bifurcation (critical) values of the parameter p were obtained (for $\delta = 0.05$): $p_{1a} \simeq 0.1085$, $p_{2a} \simeq 0.7719$, and $p_{3a} \simeq 1.958$ (problem A) and $p_{1b} \simeq 0.3915$, $p_{2b} \simeq 1.313$, and $p_{3b} \simeq 2.763$ (problem B). The corresponding values of the parameter λ are $\lambda_{1a} = 1$, $\lambda_{2a} \simeq 7.114$, $\lambda_{3a} \simeq 18.05$, $\lambda_{1b} \simeq 3.609$, $\lambda_{2b} \simeq 12.101$, and $\lambda_{3b} \simeq 25.47$.

Figure 1 shows the parameter w proportional to the deflection at the point $t = \delta$ as a function of the parameter $\lambda = p/p_{1a}$ for the first three buckling modes of the plate: curves 1a, 2a, and 3a are the branches of the solution of problem A, and curves 1b, 2b, and 3b are the branches of the solution of problem B. Since in the unbuckled (plane) states, which $w \equiv 0$, these states correspond to points on the abscissa, including bifurcation points.

The problem considered admits pairwise solutions which are symmetric about the plate plane. Therefore, Fig. 1 can be complemented by reflection with respect to the abscissa if the parameter w is defined by $w = z(\delta)/R$.

For problem A, the values of the state parameters at some points of the plate are listed in Tables 1, 2, and 3 (branches 1a, 2a, and 3a, respectively). The point numbers are given in the first columns. The first parameter $\theta(1)$ was specified in the calculations and the other parameters were calculated by the shooting method.

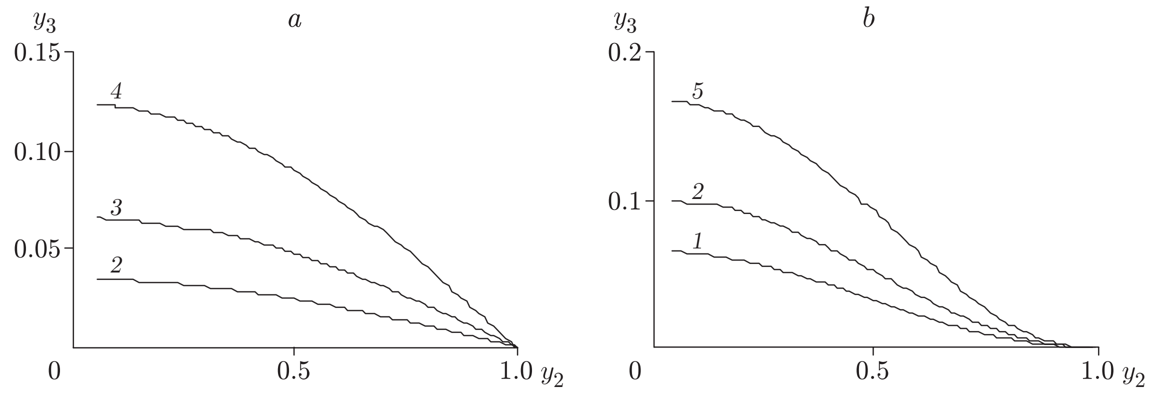

Fig. 2

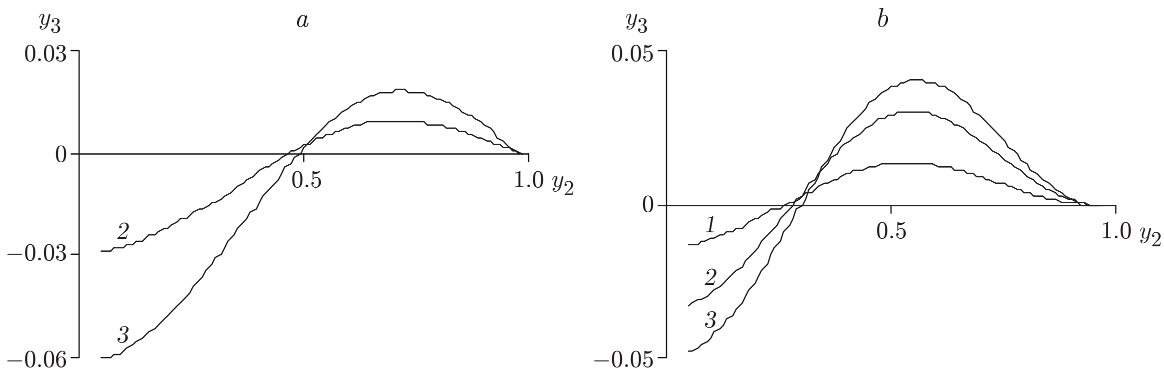

Fig. 3

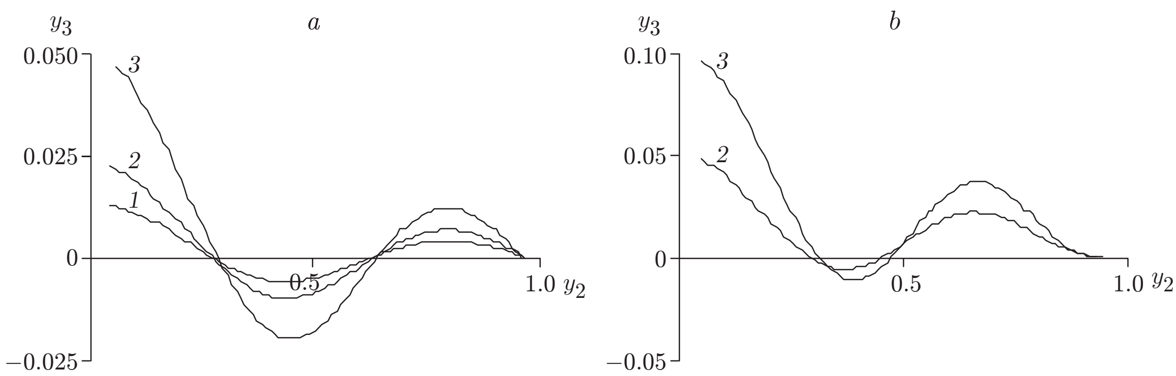

Fig. 4

TABLE 1

Point number	$\theta(1)$	λ	$10u$	w	$\tau_1(1)$	$\tau_2(1)$	$\mu(\delta)$
0	0	1	0.0203	0	−0.1085	−0.1085	0
1	0.01	1.002	0.0206	0.0068	−0.1087	−0.1095	0.0233
2	0.05	1.042	0.0260	0.0336	−0.1130	−0.1323	0.1141
3	0.10	1.162	0.0424	0.0658	−0.1255	−0.2010	0.2157
4	0.20	1.554	0.1011	0.1228	−0.1653	−0.4458	0.3622
5	0.30	2.050	0.1856	0.1704	−0.2125	−0.7954	0.4444

TABLE 2

Point number	$\theta(1)$	λ	$10u$	w	$\tau_1(1)$	$\tau_2(1)$	$\mu(\delta)$
0	0	7.114	0.1447	0	−0.7719	−0.7719	0
1	0.03	7.216	0.1491	−0.0173	−0.7826	−0.7922	−0.3031
2	0.05	7.397	0.1570	−0.0291	−0.8016	−0.8285	−0.4980
3	0.10	8.252	0.1949	−0.0607	−0.8909	−1.0022	−0.9298
4	0.20	11.448	0.3482	−0.1300	−1.2174	−1.6973	−1.4312
5	0.25	13.505	0.4578	−0.1636	−1.4197	−2.1862	−1.5216

TABLE 3

Point number	$\theta(1)$	λ	$10u$	w	$\tau_1(1)$	$\tau_2(1)$	$\mu(\delta)$
0	0	18.046	0.3672	0	−1.9580	−1.9580	0
1	0.03	18.201	0.3729	0.0130	−1.9740	−1.9850	0.5939
2	0.05	18.477	0.3832	0.0219	−2.0023	−2.0333	0.9778
3	0.10	19.791	0.4324	0.0459	−2.1365	−2.2639	1.8420
4	0.20	25.039	0.6375	0.1030	−2.6626	−3.2157	2.8734
5	0.30	32.707	0.9697	0.1635	−3.3902	−4.7262	3.9066

The corresponding data for problem B (branches 1b, 2b, and 3b) are listed in Tables 4, 5,and 6, respectively [in the calculations, the parameter $\mu(1)$ proportional to the bending moment at the boundary point was specified]. The first rows of the tables correspond to the bifurcation points lying on the abscissa axis (Fig. 1).

Figures 2–4 show the buckling modes of the plate meridian that refer to various points of branches 1, 2, and 3, respectively. Modes 2, 3, and 4 in Fig. 2a refer to the points of the first branch from Table 1 and modes 1, 2, and 5 in Fig. 2b refer to the points of the first branch from Table 4. Figure 3a and b and Fig. 4a and b show higher modes for the points of Tables 2, 5 and 3, 6, respectively. One can see that in both problems, the first modes have no internal nodes, the second modes have one node, and the third modes have two nodes. The radial displacement of the boundary point of the meridian is more pronounced in Fig. 4 since this displacement of the third modes is much larger than that of the lower modes.

The data given in the tables give an insight into the evolution of the force parameters of the plate. The maximum absolute values of the force parameters occur on the boundary contour, and the parameter τ_2 increases faster than τ_1 as λ increases. The maximum bending moments occur at the center of the plate (see the last columns of the tables). The radial and circumferential forces are compressive, nearly constant, and equal in the neighborhood of the critical points. With distance from the critical points, the difference between the magnitudes of these forces increases. The compressive forces increase near the plate edge and decrease (to zero) as the pole is approached; with further increase in the load, an extension zone forms near the pole. The monotonic variation in the forces and moments along the coordinate inherent to the first branch of solutions is violated for high values of the loading parameter. As the load increases, the oscillations of the forces and moments are enhanced and nonlinearity shows up for increasingly smaller deviations from the plane state of equilibrium.

Conclusions. The results of the numerical analysis of the nonlinear problem considered agree with theoretical findings [8–11]. The critical values of the compressive force obtained in the exact nonlinear formulation for an isotropic plate are very close (with accuracy up to three digits) to the eigenvalues of the linearized problem [2, 5]. The global solutions of the nonlinear problem are obtained by the shooting method with specified accuracy (see

TABLE 4

Point number	$\mu_1(1)$	λ	$10u$	w	$\tau_1(1)$	$\tau_2(1)$	$\mu(\delta)$
0	0	3.609	0.0734	0	−0.3915	−0.3915	0
1	−0.3	4.017	0.0945	0.0640	−0.4358	−0.4869	0.4176
2	−0.5	4.576	0.1249	0.0996	−0.4965	−0.6238	0.6104
3	−0.7	5.213	0.1615	0.1297	−0.5656	−0.7874	0.7417
4	−0.9	5.867	0.2013	0.1554	−0.6366	−0.9644	0.8306
5	−1.0	6.193	0.2219	0.1670	−0.6719	−1.0557	0.8640

TABLE 5

Point number	$\mu_1(1)$	λ	$10u$	w	$\tau_1(1)$	$\tau_2(1)$	$\mu(\delta)$
0	0	12,101	0,2462	0	−1.3130	−1.3130	0
1	−0.3	12.334	0.2553	−0.0136	−1.3382	−1.3559	−0.5914
2	−0.7	13.287	0.2932	−0.0329	−1.4417	−1.5332	−1.2573
3	−1.0	14.352	0.3362	−0.0481	−1.5572	−1.7341	−1.6213
4	−1.5	16.478	0.4251	−0.0736	−1.7879	−2.1474	−1.9948
5	−2.5	21.068	0.6330	−0.1190	−2.2859	−3.1035	−2.2428
6	−3.5	25.524	0.8569	−0.1554	−2.7693	−4.1198	−2.2419

TABLE 6

Point number	$\mu_1(1)$	λ	$10u$	w	$\tau_1(1)$	$\tau_2(1)$	$\mu(\delta)$
0	0	25.468	0.5181	0	−2.7633	−2.7633	0
1	−0.5	25.889	0.5332	0.0240	−2.8089	−2.8349	1.1428
2	−1.0	27.094	0.5765	0.0481	−2.9397	−3.0408	2.1420
3	−2.0	31.215	0.7273	0.0954	−3.3868	−3.7557	3.4299
4	−3.0	36.432	0.9274	0.1386	−3.9529	−4.6979	3.9113
5	−3.5	39.156	1.0369	0.1582	−4.2484	−5.2099	3.9788

Figs. 2–4). The parametric bifurcation conditions are represented by a planar projection (see Fig. 1). Evaluation of the potential-energy levels of the equilibrium states of the plate for a fixed loading parameter shows that the first buckling modes have the lowest level and plane states have the maximum level.

The equations derived in the paper can be used to formulate and solve strongly nonlinear axisymmetric problems for isotropic and transversely isotropic plates under different loads and boundary conditions.

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